Pergamon

0021–8928(95)00019-4

J. Appl. Maths Mechs, Vol. 59, No. 1, pp. 159–163, 1995

0021–8928/95 $24.00+0.00

THE THERMAL-STRESS STATE OF HEAT-SENSITIVE CERAMIC TUBULAR SYSTEMS IN THE CASE OF CONVECTIVE HEAT EXCHANGE†

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(*Received* 25 *October* 1993)

The thermal stresses and displacements that occur in an elastic hollow circular cylinder made of ceramics, when there is convective heat exchange are determined. The dependence of the parameters of the thermally stressed state on the radial coordinate is investigated under different heat-exchange conditions and ratios of the cylinder radii. The results obtained are used to interpret the thermoelastic behaviour of heat-sensitive ceramic tubes when the temperature conditions change.

Tubes and other cylindrical articles with an axisymmetrical temperature distribution are analysed using well-known methods [1–3]. However, there is a considerable difference between the behaviour of circular cylinders made of heat-sensitive and non-heat-sensitive materials. When there is uniform heating or cooling of an isotropic non-heat-sensitive hollow cylinder with a free surface (ignoring the temperature dependence of the thermoelastic properties) no stresses occur in it and all its dimensions change by the same amount. When the temperature changes in a heat-sensitive body its internal and external radii may both decrease and increase.

The problem of the thermoelasticity of an infinite plate of thick polycrystalline Al_2O_3, free from external forces and heated asymmetrically was derived in [4]. Here the dependence of the thermal conductivity of the material on temperature in the range 100–700°C was taken in the form [5]

$$\lambda_t(t) = a_0 + a_1 / t \tag{1}$$

where a_0 and a_1 are constants and t is the temperature of the body. When $100°C \leqslant t \leqslant 600°C$, Eq. (1) is approximated quite well by the expression

$$\lambda_t(t) = a / t, \quad a = 117,\ 1 \times 10^2 \ \text{W/m} \tag{2}$$

Under practical conditions ceramic structural components undergo heating and cooling. In order to investigate the effect of the heat sensitivity of a material, specified by (2), on the nature and value of the temperature stresses in a circular cylinder of internal radius R_1, external radius R_2 with a side surface free from stresses, subjected to convective heat exchange, we will consider the corresponding static problem of thermoelasticity.

The steady temperature field is found by solving a boundary-value problem for the heat-conduction equation

$$\frac{1}{r}\frac{d}{dr}\left[r\lambda_t(t)\frac{dt}{dr}\right] = 0 \tag{3}$$

$$r = R_1 : \lambda_t(t)\frac{dt}{dr} - \alpha_1(t - t_1) = 0 \tag{4}$$

$$r = R_2 : \lambda_t(t)\frac{dt}{dr} + \alpha_2(t - t_2) = 0$$

where α_1 and α_2 are the heat-transfer coefficients from the surfaces $r = R_1$ and $r = R_2$, respectively, and t_1 and t_2 are the temperatures of the external media surrounding these surfaces.

Integrating Eq. (3) twice and taking into account the boundary conditions (4), we obtain the solution of the heat-conduction problem in the form

$$\theta = \gamma\rho^{-\kappa} \tag{5}$$

$$\theta = \frac{t}{t_0},\ \gamma = \vartheta_0 - \frac{\kappa}{\text{Bi}_1},\ \rho = \frac{r}{R_1},\ \vartheta_0 = \frac{t_1}{t_0},\ \text{Bi}_1 = \frac{\alpha_1 R_1 t_0}{a}$$

†*Prikl. Mat. Mekh.* Vol. 59, No. 1, pp. 168–173, 1995.

where t_0 is the temperature of the unstressed and unstrained state, and κ is a dimensionless constant of integration which is the solution of the transcendental equation

$$(\mathrm{Bi}_2+\kappa)\vartheta\rho_2^{\kappa}+\kappa\varepsilon-\mathrm{Bi}_2=0$$
$$\rho_2=\frac{R_2}{R_1},\ \vartheta=\frac{t_2}{t_1},\ \mathrm{Bi}_2=\frac{\alpha_2 R_2 t_2}{a},\ \varepsilon=\frac{\mathrm{Bi}_2}{\mathrm{Bi}_1} \tag{6}$$

The thermal-stress state of the system is defined by the equations [6]

$$\sigma_{rr}=\frac{E(t)}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du}{dr}+\nu\frac{u}{r}\right]-\frac{E(t)}{1-2\nu}\Phi(t)$$
$$\sigma_{\varphi\varphi}=\frac{E(t)}{(1+\nu)(1-2\nu)}\left[\nu\frac{du}{dr}+(1-\nu)\frac{u}{r}\right]-\frac{E(t)}{1-2\nu}\Phi(t) \tag{7}$$
$$\sigma_{zz}=\nu(\sigma_{rr}+\sigma_{\varphi\varphi})-E(t)\Phi(t);\ \Phi(t)=\int_{t_0}^{t}\alpha_t(\zeta)d\zeta$$

where $\Phi(t)$ is the purely thermal strain, $E(t)$ is Young's modulus, $\alpha_t(t)$ is the temperature coefficient of linear expansions, ν is Poisson's ratio, and u is the component of the displacement vector (the radial component), which satisfies the equation

$$\frac{d}{dr}\left[E(t)\frac{du}{dr}\right]+\frac{\nu}{1-\nu}\frac{d}{dr}\left[E(t)\frac{u}{r}\right]+\frac{1-2\nu}{1-\nu}\frac{E(t)}{r}\left(\frac{du}{dr}-\frac{u}{r}\right)=\frac{1+\nu}{1-\nu}\frac{d}{dr}[E(t)\Phi(t)] \tag{8}$$

Young's modulus for an oxide ceramics, for example, is quite adequately described by the formula [7, 8]

$$E(t)=E_0/t \tag{9}$$

where E_0 is a certain constant, whereas the temperature coefficient of linear expansion in the 100–700°C temperature range is practically constant; however, to generalize the solution of the problem we will assume that it may vary as

$$\alpha_t(t)=\alpha/t,\quad \alpha=\text{const} \tag{10}$$

Using (5), (9) and (10) we can reduce the equation of equilibrium in displacements (8) to an equation which is related to Bessel's equation [9]

$$\frac{d^2u}{dr^2}+(\kappa+1)\frac{1}{r}\frac{du}{dr}-L^2\frac{u}{r^2}=f(r) \tag{11}$$

$$L^2=\frac{1-\nu(\kappa+1)}{1-\nu},\quad f(r)=\alpha\frac{1+\nu}{1-\nu}\frac{\kappa}{r}\left(\ln\frac{t}{t_0}-1\right)$$

The general solution of Eq. (11) will be sought in the form [10]

$$u=u_++u_-,\quad u_\pm=\left(C_\pm\pm\frac{1}{\mu}\int f(\xi)\xi^{k_\pm}d\xi\right)r^{1-k_\pm}$$

$$\mu=\sqrt{\kappa^2+4L^2},\quad k_\pm=(\kappa\mp\mu)/2+1$$

where C_+ and C_- are constants, determined from the boundary conditions

$$\sigma_{rr}(r=R_1)=\sigma_{rr}(r=R_2)=0.$$

After satisfying the boundary conditions using (7) for a radial displacement and the components of the thermal-stress tensor we obtain equations which, in dimensionless variables, have the form

$$U=\frac{2\nu-1}{\rho_0}\left(\frac{\rho_2^{k_-}-1}{a_+}\rho^{-k_+}-\frac{\rho_2^{k_-}-\rho_2^{\mu}}{a_-}\rho^{-k_-}\right)+(1+\nu)[\ln(\vartheta_0\gamma\rho^{-\kappa})+(1-\nu)(\kappa+2)-1]$$
$$\sigma_\rho=\frac{\rho^{\kappa}}{\vartheta_0\gamma}\left\{\frac{1}{\rho_0}[(1-\rho_2^{k_-})\rho^{-k_+}-(\rho_2^{\mu}-\rho_2^{\kappa_-})\rho^{-k_-}]+1\right\} \tag{12}$$

$$\sigma_\varphi = \sigma_\rho + \kappa\frac{\rho^\kappa}{\vartheta_0\gamma}, \quad \sigma_z = 2\nu\sigma_\rho + \frac{\rho^\kappa}{\vartheta_0\gamma}[\kappa\nu - \ln(\vartheta_0\gamma\rho^{-\kappa})] \tag{13}$$

$$\rho_0 = \rho_2^\mu - 1, \quad a_\pm = \frac{(1-\nu)(\mu\mp\kappa)\pm 2\nu}{2(1+\nu)}, \quad U = \frac{u}{\alpha R_1}$$

$$\sigma_j = \frac{\sigma_{jj}t_0}{E_0\alpha} \quad (j = \rho, \varphi, z)$$

If we assume that the thermoelastic characteristics are independent of temperature, i.e. we put

$$E(t) = E = \text{const}, \quad \alpha_t(t) = \alpha_t = \text{const} \tag{14}$$

in (9), we obtain, instead of (13)

$$\sigma_\rho = B(B^- - \rho^{-\kappa}) - 2\frac{1-\nu}{1-2\nu}$$

$$\sigma_\varphi = B(B^+ - (1-\kappa)\rho^{-\kappa}) - 2\frac{1-\nu}{1-2\nu}, \quad \sigma_z = \nu(\sigma_\rho + \sigma_\varphi) - \theta$$

$$B = \frac{\gamma}{2-\kappa}, \quad B^\pm = \frac{1-\rho_2^{2-\kappa}}{1-\rho_2^2} \pm \frac{1-\rho_2^{-\kappa}}{1-\rho_2^2}\left(\frac{\rho}{\rho_2}\right)^{-2}, \quad \sigma_j = \frac{\sigma_{jj}(1-\nu)}{E\alpha_t t_0}$$

The solution of this problem in this formulation was obtained previously in [11] assuming that a constant heat flux is given on the internal surface of the cylinder.

If, in addition to (14), we assume that the thermal conductivity is constant, the solution of the thermoelasticity problem in this case can be written as follows:

$$\theta = \vartheta_0[D(\mathrm{Bi}_1^*\ln\rho - 1) + 1]$$

$$\sigma_\rho = \frac{1}{2}\vartheta_0 D\mathrm{Bi}_1^*\left[\frac{\rho_2^2}{1-\rho_2^2}\left(\frac{1}{\rho_2} - 1\right)\ln\rho_2 - \ln\rho\right]$$

$$\sigma_\varphi = -\frac{1}{2}\vartheta_0 D\mathrm{Bi}_1^*\left[\frac{\rho_2^2}{1-\rho_2^2}\left(\frac{1}{\rho_2} + 1\right)\ln\rho_2 + \ln\rho + 1\right]$$

$$\sigma_z = \nu(\sigma_\rho + \sigma_\varphi) - \theta$$

$$D = \frac{(\vartheta - 1)K_\alpha\rho_2}{1 + \mathrm{Bi}_2^*\ln\rho_2 + K_\alpha\rho_2}, \quad K_\alpha = \frac{\alpha_2}{\alpha_1}, \quad \mathrm{Bi}_i^* = \frac{\alpha_i R_i}{\lambda_t} \quad (i = 1, 2)$$

When carrying out numerical experiments the constant κ in each case is found from Eq. (6). Values of $\kappa \times 10^4$ are given in Table 1 for some combinations of the criteria Bi_1 and Bi_2 ($\vartheta = 5$, $\vartheta_0 = 2$).

When carrying out calculations using Eqs (12) and (13) we took the following values: $\nu = 0.233$, $\vartheta_0 = 2$ and $\vartheta = 5$, which corresponds to the case when both boundary surfaces are heated. The continuous curves in all the figures correspond to the values $\mathrm{Bi}_1 = \mathrm{Bi}_2 = 1$, the dashed curves correspond to $\mathrm{Bi}_1 = 10$ and $\mathrm{Bi}_2 = 0.1$, and the dash–dot curves correspond to $\mathrm{Bi}_1 = 0.1$ and $\mathrm{Bi}_2 = 10$.

In Fig. 1 we show the results of calculations of the dimensional radial displacement. When a hollow circular heat-sensitive cylinder is heated the displacements on the external and internal surfaces have the same sign, but they are greater on the external surface. They decrease when ρ_2 increases, but they increase when the criterion Bi_1 is reduced and Bi_2 is increased; for different values of the heat-transfer coefficients the radial displacements are practically the same for all ρ_2.

In Figs 2 and 3 we show the components of the thermal-stress tensor in a heat-sensitive ceramic hollow cylinder as a function of the radial coordinate ρ. For the cases considered the axial components of the stress tensor are the

Table 1

ρ_2	Bi_1; Bi_2		
	0,1; 10	1; 1	10; 0,1
2	–2954	–5476	–787
4	–2465	–4542	–775
6	–2260	–4108	–770

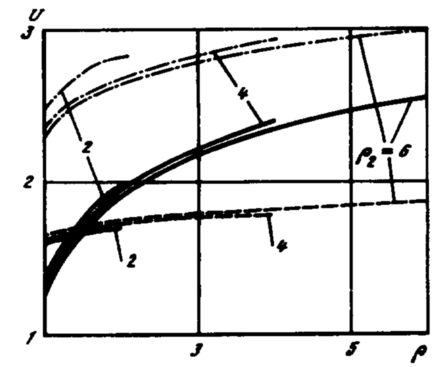

Fig. 1.

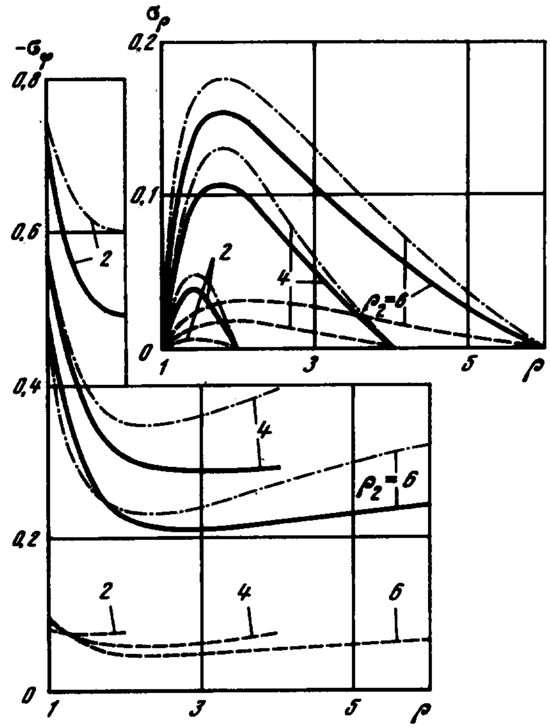

Fig. 2.

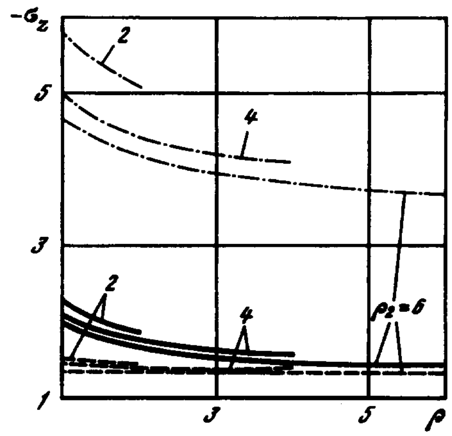

Fig. 3.

largest in absolute value, exceeding the radial and circumferential components on average by factors of 40 and 8, respectively. Whereas the radial component of the stress tensor increases as the external radius increases, the circumferential and axial components decrease, irrespective of the combinations of values of the heat-transfer coefficients from the boundary surfaces. All the components of the stress tensor increase when the heat transfer from the internal surface (Bi_1) decreases and there is a simultaneous increase in the heat transfer from the external surface (Bi_2).

The normal radial stresses are tensile stresses and, despite the fact that they are a number of times less than the values of the peripheral and axial components of the stress tensor, which are compressive, it is they that are responsible for the strength properties of the structure as a whole.

We will carry out a more detailed analysis of the stresses. As calculations show, the radial, normal and axial stresses reach an extremum, and the extremal stresses $\sigma_{\rho\max}$ (tensile) and $\sigma_{\varphi\max}$(compressive) are reached at the points $\rho = \rho_{*r}$ and $\rho = \rho_{*\varphi}$, which satisfy the following transcendental equations

$$(1-\rho_2^{k_-})\rho_{*r}^{-k_+}(\kappa-k_+)-(\rho_2^{\mu}-\rho_2^{k_-})\rho_{*r}^{-k_-}(\kappa-k_-)+\kappa\rho_0=0$$

$$(1-\rho_2^{k_-})\rho_{*\varphi}^{-k_+}(\kappa-k_+)-(\rho_2^{\mu}-\rho_2^{k_-})\rho_{*\varphi}^{-k_-}(\kappa-k_-)+\kappa\rho_0(1+\kappa)=0$$

Figure 4 illustrates the behaviour of the points of the extremum ρ_{*r} of the radial normal stresses as a function of the parameter ρ_2, which represents the ratio of the external and internal radii. The points where the radial stresses are a maximum for $\rho_2 = 2$ for all the combinations of the heat-exchange coefficients Bi_1 and Bi_2 and the values of ϑ considered lie in the range $1.33 \leqslant \rho \leqslant 1.39$. When the external radius is increased by a factor of three this range expands almost sixfold; the points of extremum themselves shift towards the external surface of the cylinder, and this displacement occurs more rapidly when $\vartheta = 0.2$ (the external surface is cooled; curves 1 in Fig. 4), then, for example, in the case when $\vartheta = 5$ (curves 2).

Note that when solving the thermoelastic problem for a circular cylinder made of a reinforced layered material, an approximate formula was established in [12] for finding the points of extremum of normal radial stresses. Unlike [12], in the case considered here the relation which enables us to find these points does not contain the values of the external and internal radii explicitly; it also contains quantities which define the heat-exchange conditions on the boundary surfaces, characterized by the parameter κ.

In Fig. 5 we show a graph of the dimensionless stress $\sigma_{\rho\max}$. The maximum radial stresses increase practically linearly as the external radius increases, and they reach a maximum level for values of the heat-exchange coefficients $Bi_1 = 0.1$ and $Bi_2 = 10$.

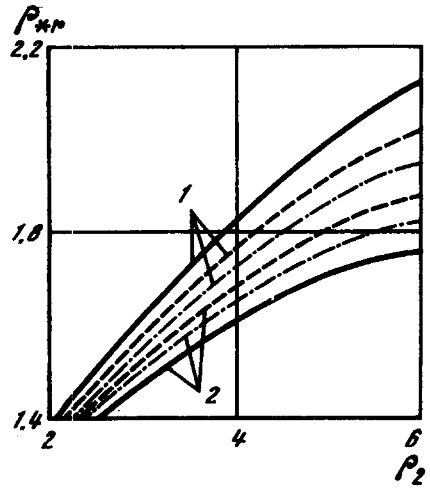

Fig. 4.

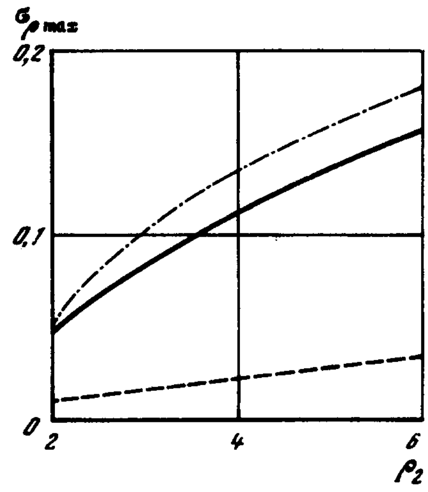

Fig. 5.

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Translated by R.C.G.